Contents

Chapter 1

Wave turbulence in magnetohydrodynamics (MHD)

This chapter reviews the recent progress made mainly during the last two decades on wave turbulence in magnetized plasmas (MHD, Hall MHD and electron MHD) in the incompressible and compressible cases. The emphasis is made on homogeneous and anisotropic turbulence which usually provides the best theoretical framework to investigate space and laboratory plasmas. The interplanetary medium and the solar atmosphere are presented as two examples of media where anisotropic wave turbulence is relevant. The most important results of wave turbulence are reported and discussed in the context of space and simulated magnetized plasmas. Important issues and possible spurious interpretations are eventually discussed.

1.1 Introduction

Wave turbulence is the study of the long-time statistical behavior of a sea of weakly nonlinear dispersive waves (Zakharov et al., 1992; Nazarenko, 2011). The energy transfer between waves occurs mostly among resonant sets of waves and the resulting energy distribution, far from a thermodynamic equilibrium, is characterized by a wide power law spectrum and a high Reynolds number. This range of wavenumbers – the inertial range – is generally localized between large scales at which energy is injected in the system (sources) and small scales at which waves break or dissipate (sinks). Pioneering works on wave turbulence date back to the sixties when it was established that the stochastic initial value problem for weakly coupled wave systems has a natural asymptotic closure induced by the dispersive nature of the waves and the large separation of linear and nonlinear time scales (Benney and Saffman, 1966; Benney and Newell, 1967, 1969). In the meantime, Zakharov and Filonenko (1966) showed that the wave kinetic equations derived from the wave turbulence analysis (with a Gaussian Ansatz applied to the four-point correlations of the wave amplitude) have exact equilibrium solutions which are the thermodynamic zero flux solutions but also – and more importantly – finite flux solutions which describe the transfer of conserved quantities between sources and sinks. The solutions, first published for isotropic turbulence (Zakharov, 1965; Zakharov and Filonenko, 1966)

were then extended to anisotropic turbulence (Kuznetsov, 1972).

Wave turbulence is a very common natural phenomenon with applications, for example, in capillary waves (Kolmakov et al., 2004; Abdurakhimov et al., 2008), gravity waves (Falcon et al., 2007), superfluid helium and processes of Bose-Einstein condensation (Kolmakov et al., 1995; Lvov et al., 2003), nonlinear optics (Dyachenko et al., 1992), inertial waves (Galtier, 2003; Morize et al., 2005) or Alfvén waves (Galtier et al., 2000; Kuznetsov, 2001; Chandran, 2005). The most important difference between plasmas and incompressible neutral fluids is the plethora of linear waves supported by the former. The direct consequence is that in weakly nonlinear plasmas the fundamental entities are waves rather than the eddies of strong turbulence (Kolmogorov, 1941; Krommes, 2002). In the situation discussed here – magnetized plasmas seen as fluids – wave and strong turbulence may coexist and therefore both waves and eddies have in fact an impact on the nonlinear dynamics which is strongly anisotropic. Anisotropic turbulence is particularly well observed in space plasmas since a magnetic field is often present on the largest scale of the system, like in the inner interplanetary medium where the magnetic field lines form an Archimedean spiral near the equatorial plane (Goldstein and Roberts, 1999), at the solar surface where coronal loops and open magnetic flux tubes are found (Cranmer et al., 2007) or in planetary magnetospheres where shocks and discontinuities are measured (Sahraoui et al., 2006).

In the present chapter, a review is made on wave turbulence in magnetized plasmas. A plasma is a gaseous state of matter in which the atoms or molecules are strongly ionized. Mutual electromagnetic forces between the ions and the free electrons are then playing dominant roles which adds to the complexity as compared to the situation in neutral atomic or molecular gases. In the situation discussed here the plasmas are described by the magnetohydrodynamics (MHD) approximation in the incompressible or compressible case. The role played by the Hall term is discussed through the Hall MHD description as well as its small-scale limit of electron MHD.

The structure of the chapter is as follows. Physical motivations for developing wave turbulence theories are given in Section 1.2 where we first describe multiscale solar wind turbulence, and then present the coronal heating problem. Section 1.3 emphasizes the differences between strong and wave turbulence in MHD and the path followed historically by researches to finally obtain the MHD wave turbulence theories. In Section 1.4, the wave turbulence formalism is exposed with the basic ideas to derive the wave kinetic equations. Section 1.5 deals with the results obtained under different approximations (MHD, Hall MHD and electron MHD) in the incompressible or compressible case. Finally we conclude with a general discussion in the last Section.

1.2 Waves and turbulence in space plasmas

Waves and turbulence are ubiquitous in astrophysical plasmas. Their signatures are found in the Earth's magnetosphere (Sahraoui et al., 2006), the solar corona (Chae et al., 1998), the solar wind (Bruno and Carbone, 2005) or the interstellar medium (Elmegreen and Scalo, 2004; Scalo and Elmegreen, 2004). These regions are characterized by extremely large (magnetic) Reynolds numbers, up to 10^{13}, with a range of available scales from 10^{18}m to a few meters.

1.2.1 *Interplanetary medium*

Extensive investigations are made in the interplanetary medium (and in the Earth's magnetosphere which is not the subject discussed here) where many *in situ* spacecrafts measurements are available. The solar wind plasma is found to be in a highly turbulent state with magnetic and velocity fluctuations detected from 10^{-6}Hz up to several hundred Hz (Coleman, 1968; Roberts et al., 1987; Leamon et al., 1998; Smith et al., 2006). The turbulent state of the solar wind was first suggested in 1968 (Coleman, 1968) when a power law behavior was reported for energy spectra with spectral indices lying between -1 and -2 (with the use of the Taylor "frozen-in flow" hypothesis). More precise measurements revealed that the spectral index at low frequency ($<$ 1Hz) is often about -1.7 which is closer to the Kolmogorov prediction (Kolmogorov, 1941) for neutral fluids ($-5/3$) rather than the Iroshnikov-Kraichnan prediction (Iroshnikov, 1964; Kraichnan, 1965) for magnetized fluids ($-3/2$). Alfvén waves are also well observed since 1971 (Belcher and Davis, 1971) with a strong domination of antisunward propagative waves at short heliocentric distances (less than 1 AU). Since pure (plane) Alfvén waves are exact solutions of the ideal incompressible MHD equations (see e.g., Pouquet, 1993), nonlinear interactions should be suppressed if only one type of waves is present. Therefore sunward Alfvén waves, although subdominant, play an important role in the internal solar wind dynamics.

The variance analysis of the magnetic field components and of its magnitude shows clearly that the magnetic field vector of the (polar) solar wind has a varying direction but with only a weak variation in magnitude (Forsyth et al., 1996). Typical values give a normalized variance of the field magnitude smaller than 10% whereas for the components it can be as large as 50%. In these respects, the inner interplanetary magnetic field may be seen as a vector lying approximately around an Archimedean spiral direction with only weak magnitude variations (Barnes, 1981). Solar wind anisotropy with more power perpendicular to the mean magnetic field than that parallel to it, is pointed out by data analysis (Klein et al., 1993) that provides a ratio of power up to 30. From single-point spacecraft measurements it is not possible to specify the exact three-dimensional form of the spectral tensor of the magnetic or velocity fluctuations. However, it is possible to show that the spacecraft-frame spectrum may depend on the angle between the local magnetic

field direction and the flow direction (Horbury et al., 2008). In the absence of such data, a quasi two-dimensional model was proposed (Bieber et al., 1996) in which wave vectors are nearly perpendicular to the large-scale magnetic field. It is found that about 85% of solar wind turbulence possesses a dominant 2D component. Additionally, solar wind anisotropies is detected through radio wave scintillations which reveal that density spectra close to the Sun are highly anisotropic with irregularities stretched out mainly along the radial direction (Armstrong et al., 1990). More recently

For frequencies larger than 1Hz, a steepening of the magnetic fluctuation power law spectra is observed over more than two decades (Coroniti et al., 1982; Denskat et al., 1983; Leamon et al., 1998; Bale et al., 2005; Smith et al., 2006) with a spectral index close to -2.5. This new inertial range seems to be characterized by a bias of the polarization suggesting that these fluctuations are likely to be right-hand polarized, outward propagating waves (Goldstein et al., 1994). Various indirect lines of evidence indicate that these waves propagate at large angles to the background magnetic field and that the power in fluctuations parallel to the background magnetic field is still less than the perpendicular one (Coroniti et al., 1982; Leamon et al., 1998). For these reasons, it is thought (Stawicki et al., 2001) that Alfvén – left circularly polarized – fluctuations are suppressed by proton cyclotron damping and that the high frequency power law spectra are likely to consist of whistler waves. This scenario is supported by multi-dimensional direct numerical simulations of compressible Hall MHD turbulence in the presence of an ambient field (Ghosh et al., 1996) where a steepening of the spectra was found on a narrow range of wavenumbers, and associated with the appearance of right circularly polarized fluctuations. This result has been confirmed numerically with a turbulent cascade (shell) model based on 3D Hall MHD in which a well extended steeper power law spectrum was found at scale smaller than the ion skin depth (Galtier and Buchlin, 2007). (Note that in this cascade model no mean magnetic field is assumed.) However, the exact origin of the change of statistical behavior is still under debate (Markovskii et al., 2008): for example, an origin from compressible effects is possible in the context of Hall MHD (Alexandrova et al., 2008); a gyrokinetic description is also proposed in which kinetic Alfvén waves play a central role (Howes et al., 2008).

The solar wind plasma is currently the subject of a new extensive research around the origin of the spectral break observed in the magnetic fluctuations accompanied by an the absence of intermittency (Kiyani et al., 2009). We will see that wave turbulence may have a central role in the sense that it is a useful point of departure for understanding the detailed physics of solar wind turbulence. In particular, it gives strong results in regards to the possible multiscale behavior of magnetized plasmas as well as the intensity of the anisotropic transfer between modes.

1.2.2 *Solar atmosphere*

Although it is not easy to measure directly the coronal magnetic field, it is now commonly accepted that the structure of the low solar corona is mainly due to the magnetic field (Aschwanden et al., 2001). The high level of coronal activity appears through a perpetual impulsive reorganization of magnetic structures over a large range of scales, from about 10^5 km until the limit of resolution, about one arcsec ($<$ 726 km). The origin of the coronal reorganization is currently widely studied in solar physics. Information about the solar corona comes from spacecraft missions like SoHO/ESA or TRACE/NASA launched in the 1990s, or from more recent spacecrafts STEREO/NASA, Hinode/JAXA or SDO/NASA (see Fig. 1.1). The most recent observations reveal that coronal loops are not yet resolved transversely and have to be seen as tubes made of a set of strands which radiate alternatively. In fact, it is very likely that structures at much smaller scales exist but have not yet been detected (see e.g., Warren, 2006).

Observations in UV and X-ray show a solar corona extremely hot with temperatures exceeding 10^6K – close to hundred times the solar surface temperature. These coronal temperatures are highly inhomogeneous: in the quiet corona much of the plasma lies near 1–2 $\times$ 10^6K and 1–8 $\times$ 10^6K in active regions. Then, one of the major questions in solar physics concerns the origin of such high values of coronal temperature. The energy available in the photosphere – characterized by granules – is clearly sufficient to supply the total coronal losses (Priest, 1982) which is about $10^4 \mathrm{J\,m^{-2}s^{-1}}$ for active regions and about one or two orders of magnitude smaller for the quiet corona and coronal holes where open magnetic field lines emerge. The main issue is thus to understand how the available photospheric energy is transferred and accumulated in the solar corona, and by what processes it is dissipated.

In active region loops, analyses made by spectrometers show that the plasma velocity can reach values up to 50 km/s (Brekke et al., 1997). The highly dynamical nature of some coronal loops is also pointed out by non-thermal velocities reaching sometimes 50 km/s as it was revealed for example by SoHO/ESA (Chae et al., 1998). These observations give also evidences that the line broadening is due to motions which are still not resolved neither in space, with scales smaller than the diameter of coronal loops, nor in time, with timescales shorter than the exposure time of the order of few seconds. These velocity measurements are very often interpreted as a signature of MHD turbulence where small scales are produced naturally *via* a nonlinear cascade of energy. In the light of the most recent observations, it seems fundamental to study, both theoretically and numerically, the impact of small-scale phenomena on the coronal heating. Note that recent Hinode/JAXA and SDO/NASA pictures seem to show a magnetic field controlled by plasma turbulence at all scales in which Alfvén waves are omnipresent (see e.g., Doschek et al., 2007; Nishizuka et al., 2008; Cargill and De Moortel, 2011). Thus, the turbulent activity of the corona is one of the key issues to understand the heating processes.

In the framework of turbulence, the energy supplied by the photospheric motions

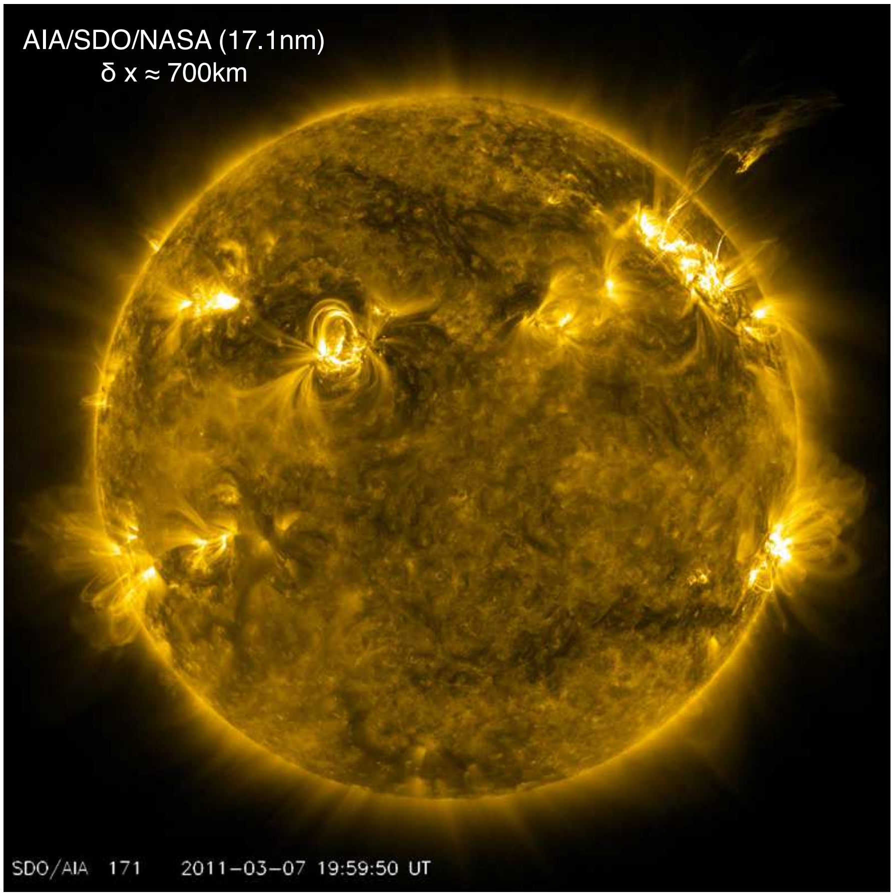

Fig. 1.1 Solar corona seen at 17.1nm by the AIA instrument onboard SDO (Solar Dynamics Observatory) – NASA credits. The spatial resolution of the picture is about 700km.

and transported by Alfvén waves through the corona is transferred towards smaller and smaller scales by nonlinear coupling between modes (the so-called energy cascade) until dissipative scales are reached from which the energy is converted into heating. The main coronal structures considered in such a scenario are the magnetic loops which cover the solar surface. Each loop is basically an anisotropic bipolar structure anchored in the photosphere. It forms a tube of magnetic fields in which the dense and hot matter is confined. Because a strong guiding magnetic field ($\mathbf{B_0}$) is present, the nonlinear cascade that occurs is strongly anisotropic with small scales mainly developed in the $\mathbf{B_0}$ transverse planes. Most of the models published deals with isotropic MHD turbulence (see e.g., Hendrix and Van Hoven, 1996) and it is only very recently that anisotropy has been included in turbulent heating models (Buchlin and Velli, 2007).

The latest observations show that waves and turbulence are among the main ingredients of the solar coronal activity. Weak MHD turbulence is now invoked has a possible regime for some coronal loops since a very small ratio is found between the fluctuating magnetic field and the axial component (Rappazzo et al., 2007, 2008). Inspired by the observations and by recent direct numerical simulations of 3D MHD turbulence (Bigot et al., 2008b), an analytical model of coronal structures has been proposed (Bigot et al., 2008c) where the heating is seen as the end product of a wave turbulent cascade. Surprisingly, the heating rate found is non negligible and may explain the observational predictions.

The coronal heating problem also concerns the regions where the fast solar wind is produced, *i.e.* the coronal holes (Hollweg and Isenberg, 2002; Cranmer et al., 2007). Observations seem to show that the heating affects preferentially the ions in the direction perpendicular to the mean magnetic field. The electrons are much cooler than the ions, with temperatures generally less than or close to 10^6K (see *e.g.*, David et al., 1998). Additionally, the heavy ions become hotter than the protons within a solar radius of the coronal base. Ion cyclotron waves could be the agent which heats the coronal ions and accelerates the fast wind. Naturally the question of the origin of these high frequency waves arises. Among different scenarios, turbulence appears to be a natural and efficient mechanism to produce ion cyclotron waves. In this case, the Alfvén waves launched at low altitude with frequencies in the MHD range, would develop a turbulent cascade to finally degenerate and produce ion cyclotron waves at much higher frequencies. In that context, the wave turbulence regime was considered in the weakly compressible MHD case at low-β plasmas (where β is the ratio between the thermal and magnetic pressure) in order to analyze the nonlinear three-wave interaction transfer to high frequency waves (Chandran, 2005). The wave turbulence calculation shows – in absence of slow magnetosonic waves – that MHD turbulence provides a convincing explanation for the anisotropic ion heating.

1.3 Turbulence and anisotropy

This Section is first devoted to the comparison between wave and strong turbulence. In particular, we will see how the theoretical questions addressed at the end of the 20th century have led to the emergence of a large number of papers on wave turbulence in magnetized plasmas and to many efforts to characterize the fundamental role of anisotropy.

1.3.1 *Navier-Stokes turbulence*

Navier-Stokes turbulence is basically a strong turbulence problem in which it is impossible to perform a (non trivial and) consistent linearization of the equations against a stationary homogeneous background. We remind that wave tur-

bulence demands the existence of linear (dispersive) propagative waves as well as a large separation of linear and nonlinear (eddy-turnover) time scales (see *e.g.*, Benney and Newell, 1969). In his third 1941 turbulence paper, Kolmogorov (1941) found that an exact and nontrivial relation may be derived from Navier-Stokes equations for the third-order longitudinal structure function (Kolmogorov, 1941): it is the so-called 4/5s law. Because of the rarity of such results, the Kolmogorov's 4/5s law is considered as one of the most important results in turbulence (Frisch, 1995). Basically, this law makes the following link in the 3D physical space between a two-point measurement, separated by a distance $\mathbf{r}$, and the distance itself[1]

$$-\frac{4}{5}\varepsilon^v r = \langle (v'_L - v_L)^3 \rangle \,, \tag{1.1}$$

where $\langle\rangle$ denotes an ensemble average, the longitudinal direction L is the one along the vector separation $\mathbf{r}$, v is the velocity and ε^v is the mean (kinetic) energy injection or dissipation rate per unit mass. To obtain this exact relation, the assumptions of homogeneity and isotropy are made (Batchelor, 1953). The former assumption is satisfied as long as we are at the heart of the fluid (far from the boundaries) and the latter is also satisfied if no external agent (like, for example, rotation or stratification) is present. Additionally, the long time limit is considered for which a stationary state is reached with a finite ε^v and the infinite Reynolds number limit (*i.e.* the viscosity $\nu \to 0$) is taken for which the mean energy dissipation rate per unit mass tends to a finite positive limit. Therefore, the exact prediction is valid in the asymptotic limit of a large inertial range. The Kolmogorov law is well supported by the experimental data (see *e.g.*, Frisch, 1995).

The 4/5s law is a fundamental result used to develop heuristic spectral scaling laws like the famous – but not exact – 5/3-Kolmogorov energy spectrum. This point makes a fundamental difference with wave turbulence where the power law spectra found are exact solutions of the asymptotically exact wave turbulence equations. Nevertheless, the term "Kolmogorov theory" is often associated to the $-5/3$ spectrum since there exists a theory behind in the physical space.

1.3.2 *Incompressible MHD Turbulence*

1.3.2.1 *Strong turbulence*

The wave turbulence regime exists in incompressible MHD. The main reason is that Alfvén waves are linear solutions when a stationary homogeneous background magnetic field $\mathbf{B_0}$ is applied. This statement seems to be obvious but we will see that the problem is subtle and the existence of an Alfvén wave turbulence theory was the subject of many discussions basically because those waves are only pseudo-dispersive (*i.e.* , the frequency ω is proportional to a wavenumber).

[1] The Kolmogorov 4/5s law can be written in an equivalent way as a 4/3s law (Antonia et al., 1997; Rasmussen, 1999), namely: $-(4/3)\varepsilon^v r = \langle (v'_L - v_L) \sum_i (v'_i - v_i)^2 \rangle$.

The question of the existence of an exact relation for third order structure functions is naturally addressed for strong (without $\mathbf{B_0}$) MHD turbulence. The answer was given by Politano & Pouquet only in 1998 (see also, Chandrasekhar, 1951) for incompressible MHD turbulence. The presence of the magnetic field and its coupling with the velocity field renders the problem more difficult and, in practice, we are dealing with a couple of equations. In this case, the possible formulation in 3D is the 4/3's law

$$-\frac{4}{3}\varepsilon^{\pm} r = \langle (z'^{\mp}_L - z^{\mp}_L) \sum_i (z'^{\pm}_i - z^{\pm}_i)^2 \rangle \,, \tag{1.2}$$

where the direction L is still the one along the vector separation $\mathbf{r}$, $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$ is the Elsässer fields (with $\mathbf{b}$ normalized to a velocity field) and $\varepsilon^{\pm}$ is the mean energy dissipation rate per unit mass associated to the Elsässer energies. To obtain these exact relations, the assumptions of homogeneity and isotropy are still made, and the long time limit for which a stationary state is reached with a finite $\varepsilon^{\pm}$ is also considered. The infinite kinetic and magnetic Reynolds number limit ($\nu \to 0$ and the magnetic diffusivity $\eta \to 0$) for which the mean energy dissipation rates per unit mass have a finite positive limit is eventually taken. Therefore, the exact prediction is again valid, at first order, in a wide inertial range. This prediction is currently often used in the literature to analyze space plasma data and determine the local heating rate $\varepsilon^{\pm}$ in the solar wind (see *e.g.*, Sorriso-Valvo et al., 2007; MacBride et al., 2008).

1.3.2.2 *Iroshnikov–Kraichnan spectrum*

The isotropy assumption used to derive the 4/3's law is stronger for magnetized than neutral fluids since in most of the situations encountered in astrophysics a large scale magnetic field $\mathbf{B_0}$ is present which leads to anisotropy (see Section 1.2). Although this law is a fundamental result that may be used to develop a heuristic spectral scaling law, the role of B_0 has to be clarified. Indeed, we have now two time-scales: the eddy-turnover time and the Alfvén time. The former is similar to the eddy-turnover time in Navier-Stokes turbulence and may be associated to the distortion of wave packets (the basic entity in MHD), whereas the latter may be seen as the duration of interaction between two counter-propagating Alfvén wave packets (see Fig.1.2). During a collision, there is a deformation of the wave packets in such a way that energy is transferred mainly at smaller scales. The multiplicity of collisions leads to the formation of a well extended power law energy spectrum whose index lies between $-5/3$ (Kolmogorov prediction) and $-3/2$ (Iroshnikov-Kraichnan prediction) according to the phenomenology used, *i.e.* with or without the Alfvén wave effect. Note the slightly different approaches followed by Iroshnikov and Kraichnan to derive the $-3/2$ spectrum. In the former case the presence of a strong magnetic field is explicitly assumed whereas it is not in the latter case where it is claimed that the small-scale fluctuations see the large-scales – in the

sub-inertial range – as a spatially uniform magnetic field. However, they both assumed isotropy to finally proposed the so-called Iroshnikov-Kraichnan spectrum for MHD turbulence. It is important to note that the exact isotropic relation (1.2) in physical space corresponds dimensionally to a $-5/3$ energy spectrum since it is cubic in z (or in v and b) and linear in r. It is therefore less justified to use the term "theory" for the Iroshnikov-Kraichnan spectrum than for the Kolmogorov one for neutral fluids.

1.3.2.3 *Breakdown of isotropy*

The weakness of the Iroshnikov-Kraichnan phenomenology is the apparent contradiction between the (in)direct presence of a strong uniform magnetic field and the assumption of isotropy. An important difference between neutral and magnetized fluids is the impossibility in the latter case to remove a large-scale (magnetic) field by a Galilean transform. The role of a uniform magnetic field has been widely discussed in the literature and, in particular, during the last three decades (see *e.g.*, Montgomery and Turner, 1981; Shebalin et al., 1983; Matthaeus et al., 1996; Ng and Bhattacharjee, 1996; Verma, 2004). At strong $\mathbf{B_0}$ intensity, one of the most clearly established results is the bi-dimensionalization of MHD turbulent flows with a strong reduction of nonlinear transfers along $\mathbf{B_0}$. The consequence is an energy concentration near the plane $\mathbf{k} \cdot \mathbf{B_0} = 0$, a result illustrated later on by direct numerical simulations in two and three space dimensions (Shebalin et al., 1983).

The effects of a strong uniform magnetic field may be handled through an analysis of resonant triadic interactions (Shebalin et al., 1983) between the wavevectors $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ which satisfy the vectorial relation

$$\mathbf{k} = \mathbf{p} + \mathbf{q}\,, \tag{1.3}$$

whereas the associated wave frequencies satisfy the scalar relation

$$\omega(\mathbf{k}) = \omega(\mathbf{p}) + \omega(\mathbf{q})\,. \tag{1.4}$$

For incompressible MHD, the Alfvén frequency is

$$\omega(\mathbf{k}) = \pm \mathbf{k} \cdot \mathbf{B_0} = \pm k_\parallel B_0\,, \tag{1.5}$$

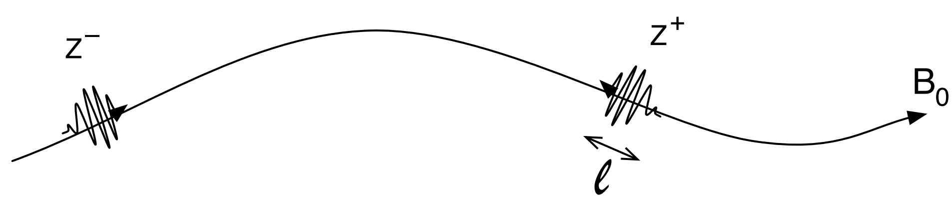

Fig. 1.2 Alfvén wave packets propagating along a large scale magnetic field $\mathbf{B_0}$.

where $\|$ defines the direction along $\mathbf{B_0}$ ($\perp$ will be the perpendicular direction to $\mathbf{B_0}$ which is written in velocity unit). The solution of the three-wave resonance condition gives for example, $q_\| = 0$, which implies a spectral transfer only in the perpendicular direction. For a strength of B_0 well above the *r.m.s.* level of the kinetic and magnetic fluctuations, the nonlinear interactions of Alfvén wave packets may dominate the dynamics of the MHD flow leading to the regime of wave turbulence (see Section 1.4) where the energy transfer, stemming from three-wave resonant interactions, can only increase the perpendicular component of the wavevectors, while the nonlinear transfers are completely inhibited along $\mathbf{B_0}$. The end result is a strongly anisotropic flow.

1.3.2.4 *Emergence of anisotropic laws*

An important issue discussed in the literature is the relationship between perpendicular and parallel scales in anisotropic MHD turbulence (see *e.g.*, Higdon, 1984; Goldreich and Sridhar, 1995; Boldyrev, 2006; Sridhar, 2010). In order to take into account anisotropy, Goldreich and Sridhar (1995) proposed a heuristic model based on the conjecture of a critical balance between the Alfvén and the eddy-turnover times, which are respectively

$$\tau_A \sim \ell_\| / B_0 \tag{1.6}$$

and

$$\tau_{eddy} \sim \ell_\perp / u_\ell \,, \tag{1.7}$$

where $\ell_\|$ and $\ell_\perp$ are typical length scales parallel and perpendicular to $\mathbf{B_0}$. The conjecture says that in the inertial range we have, $\tau_A = \tau_{eddy}$. The latter relation leads trivially to, $u_\ell \sim B_0 \ell_\perp / \ell_\|$. Following the Kolmogorov arguments, one ends up trivially with the Kolmogorov energy spectrum

$$E(k_\perp, k_\|) \sim k_\perp^{-5/3} k_\|^{-1} \,, \tag{1.8}$$

(where $\mathbf{k} \equiv (\mathbf{k}_\perp,\, k_\|)$, $k_\perp \equiv |\mathbf{k}_\perp|$, $\int\int E(k_\perp, k_\|) dk_\perp dk_\| = \int E(k_\perp) dk_\perp$), and with the non trivial anisotropic scaling law (with $u_\ell^2 / \tau_{eddy} = \text{constant}$)

$$k_\| \sim k_\perp^{2/3} \,. \tag{1.9}$$

This heuristic prediction means that anisotropy is stronger at smaller scales.

A generalization of this result was proposed by Galtier et al. (2005) in order to model MHD flows both in the wave and strong turbulence regimes, as well as for the transition between them. In this heuristic model, the constant time-scale ratio $\chi = \tau_A / \tau_{eddy}$ is not assumed to be necessarily equal to unity. The relaxation of the constraint ($\chi = 1$) allows us to recover the anisotropic scaling law (1.9) which now includes B_0,

$$k_\| \sim k_\perp^{2/3} / B_0 \,, \tag{1.10}$$

and to find a universal prediction for the total energy spectrum

$$E(k_\perp, k_\parallel) \sim k_\perp^{-\alpha} k_\parallel^{-\beta}\,, \tag{1.11}$$

with

$$3\alpha + 2\beta = 7\,. \tag{1.12}$$

Note that a classical calculation with a transfer time $\tau = \tau_{eddy}^2/\tau_A = \tau_{eddy}/\chi$ leads trivially to the spectrum (1.8): the choice of τ fixes irreversibly the spectrum. It is only when the ansatz (1.11) is introduced that the non trivial relation (1.12) emerges; this ansatz is believed to be weak since power laws are the natural solutions of the problem. According to direct numerical simulations (see *e.g.*, Cho and Vishniac, 2000; Maron and Goldreich, 2001; Ng et al., 2003; Shaikh and Zank, 2007; Bigot et al., 2008b), the anisotropic scaling law between parallel and perpendicular scales (1.9) seems to be a robust result and an approximately constant ratio χ, generally smaller than one, is found between the Alfvén and the eddy-turnover times. This sub-critical value of χ implies therefore a dynamics mainly driven by Alfvén waves interactions. Note that the presence of B_0 in relation (1.10) shows the convergence towards wave turbulence ($B_0 \to +\infty$, with respect to the fluctuations) for which the parallel transfer is totally frozen.

The question of the spectral indices is still a challenging problem in anisotropic turbulence (Sagaut and Cambon, 2008). The main conclusion of Bigot et al. (2008b) is that the difficulty to make the measurements is generally underestimated in a sense that the scaling prediction in $k_\perp$ may change significantly when $E(k_\perp, k_\parallel)$ is plotted at a given $k_\parallel$ instead of $E(k_\perp)$: indeed, the slow mode $E(k_\perp, k_\parallel = 0)$ may play a singular role in the dynamics with a scaling in $k_\perp$ different from the one given by the 3D modes $E(k_\perp, k_\parallel > 0)$. This comment holds primarily for direct numerical simulations where technically it is currently possible to make this distinction, observations being still far from this possibility. This point will be further discussed in the last Section. Note finally that all these spectral predictions suffer from rigorous justifications and the word ”theory” that we find very often in the literature is not justified at all. A breakthrough could be achieved if one could develop the equivalent of an exact 4/5's law for anisotropic MHD turbulence. Then, a dimensional derivation from this law could lead to an anisotropic spectral prediction and in some sense, for the first time, the possibility of having a theoretical link between strong and wave turbulence. Recently, an attempt has been made in that direction by using the idea of critical balance for third-order moments (Galtier, 2012). Moreover, it was shown that the introduction of the dynamic alignment conjecture into the exact relation for third-order moments (Politano and Pouquet, 1998) may give a $k_\perp^{-3/2}$ spectrum (Boldyrev et al., 2009).

1.3.3 *Towards an Alfvén wave turbulence theory*

In view of the importance of anisotropy in natural magnetized plasma (see Section 1.2), Sridhar and Goldreich (1994) suggested that a plasma evolving in a medium

permeated by a strong uniform magnetic field and in the regime of wave turbulence is characterized by four-wave nonlinear interactions. The essence of wave turbulence is the statistical study of large ensembles of weakly interacting waves via a systematic asymptotic expansion in powers of small nonlinearity. This technique leads finally to the exact derivation of wave kinetic equations for the invariants of the system like the energy spectrum (see Section 1.4). In MHD, it is the presence of a strong uniform magnetic field $\mathbf{B_0}$ that allows to introduce a small parameter in the system, namely the ratio between the fluctuating fields and B_0. The result found by Sridhar and Goldreich in 1994 implies that the asymptotic development has no solution at the main order and that we need to go to the next order to describe Alfvén wave turbulence. Several articles, using a phenomenology (see *e.g.*, Montgomery and Matthaeus, 1995; Verma, 2004) or a rigorous treatment (Ng and Bhattacharjee, 1996), were published to contest this conclusion and sustain the non trivial character of the three-wave interactions. In response, a detailed theory was finally given in 2000 (Galtier et al., 2000; Nazarenko et al., 2001; Galtier et al., 2002) whose main prediction may be derived heuristically in few lines as follows. According to Fig. 1.2 the main process which happens in Alfvén wave turbulence is the stochastic collisions of wavepackets. To find the transfer time and then the energy spectrum, first we shall evaluate the modification of a wavepacket produced by one collision. We have (for simplicity we only consider the balanced case for which $z_\ell^\pm \sim z_\ell \sim u_\ell \sim b_\ell$)

$$z_\ell(t+\tau_A) \sim z_\ell(t) + \tau_A \frac{\partial z_\ell}{\partial t} \sim z_\ell(t) + \tau_A \frac{z_\ell^2}{\ell_\perp}\,, \tag{1.13}$$

where τ_A is the duration of one collision; in other words, after one collision the distortion of a wavepacket is $\Delta_1 z_\ell \sim \tau_A z_\ell^2/\ell_\perp$. This distortion is going to increase with time in such a way that after N stochastic collisions the cumulative effect may be evaluated like a random walk

$$\sum_{i=1}^{N} \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell_\perp} \sqrt{\frac{t}{\tau_A}}\,. \tag{1.14}$$

The transfer time τ_{tr} that we are looking for is the one for which the cumulative distortion is of the order of one, *i.e.* of the order of the wavepacket itself

$$z_\ell \sim \tau_A \frac{z_\ell^2}{\ell_\perp} \sqrt{\frac{\tau_{tr}}{\tau_A}}\,, \tag{1.15}$$

then we obtain

$$\tau_{tr} \sim \frac{1}{\tau_A} \frac{\ell_\perp^2}{z_\ell^2} \sim \frac{\tau_{eddy}^2}{\tau_A}\,. \tag{1.16}$$

A classical calculation, with $\varepsilon \sim z_\ell^2/\tau_{tr}$, leads finally to the energy spectrum

$$E(k_\perp, k_\parallel) \sim \sqrt{\varepsilon B_0}\, k_\perp^{-2} k_\parallel^{-1/2}\,, \tag{1.17}$$

where $k_\parallel$ has to be seen as a parameter since no transfer along the parallel direction is expected (see Section 1.3.2.3). Note that this demonstration is the one traditionally used for deriving the Iroshnikov-Kraichnan spectrum, but since in this case isotropy is assumed a $E(k) \sim \sqrt{\varepsilon B_0}\, k^{-3/2}$ is predicted.

First signatures that may be attributed to wave turbulence were found by Perez and Boldyrev (2008) in numerical simulations of a reduced form of the MHD equations (Galtier and Chandran, 2006). The detection of the wave turbulence regime from direct numerical simulations of MHD equations is still a difficult task but recent results have been obtained in which temporal, structural and spectral signatures are reported (Bigot et al., 2008a,b; Bigot and Galtier, 2011). Current efforts are also made to analyze the effects of other inviscid invariants, like the cross-helicity, on the scaling laws of wave turbulence (Lithwick and Goldreich, 2003; Chandran, 2008). It is worth noting that these works on imbalanced wave turbulence have been followed by (sometimes controversial) research investigations in the strong turbulence regime which is of great relevance for the solar wind (see *e.g.* Chandran et al., 2009; Beresnyak and Lazarian, 2010; Podesta and Bhattacharjee, 2010).

1.3.4 *Wave turbulence in compressible MHD*

Most of the investigations devoted to wave turbulence refers to isotropic media where the well known conformal transform proposed by Zakharov and Filonenko (1966) may be applied to find the so-called Kolmogorov-Zakharov spectra (see Section 1.4.4). (Surprisingly a similar transform was used in the meantime by Kraichnan (1967) to investigate the problem of 2D turbulence.) The introduction of anisotropy in plasmas was studied to a smaller extent: the first example is given by magnetized ion-sound waves (Kuznetsov, 1972). The compressible MHD case was analyzed later by Kuznetsov (2001) for a situation where the plasma (thermal) pressure is small compared with the magnetic pressure (small β limit). In this case, the main nonlinear interaction involving MHD waves is the scattering of a fast magneto-acoustic and Alfvén waves on slow magneto-acoustic waves. In this process, the fast and Alfvén waves act as high-frequency waves with respect to the slow waves. (To simplify the analysis other three-wave interaction processes that do not involve slow waves are neglected.)

A variant of the wave turbulence analysis in compressible MHD was proposed by Chandran (2005) in the limit of a small β in which the slow waves are neglected and a constant density is imposed. The other (mathematical) difference is that the Hamiltonian formalism was used in the former analysis whereas an Eulerian description was employed in the latter case. Because the compressible regime is much more difficult to analyze than the incompressible one, simplifications have been made and to date, no general theory has been proposed for compressible MHD wave turbulence.

1.3.5 *Wave turbulence in Hall and electron MHD*

Modeling the physics of a plasma beyond the MHD approximation, namely on spatial scales shorter than the ion inertial length d_i (but larger than the electron inertial length d_e) and time scales shorter than the ion cyclotron period $1/\omega_{ci}$, is a highly challenging problem even in the fluid approximation. (For kinetic models we can mention the gyrokinetic approximation useful for weakly collisional plasmas in which time scales are supposed to be much larger than $1/\omega_{ci}$ (see *e.g.*, Schekochihin et al., 2009).) In that context the electron MHD approximation (Kingsep et al., 1990) is often used: in such a limit one assumes that ions do not have time to follow electrons and provide a static homogeneous background on which electrons move. The electron MHD approximation is particularly relevant in the context of collisionless magnetic reconnection where a diffusion region is developed with multiscale structures corresponding to ion and electron characteristic lengths (Biskamp, 1997; Bhattacharjee, 2004; Yamada, 2007).

An important issue in electron MHD turbulence is about the impact of whistler waves on the dynamics. Biskamp et al. (1999) argued that although whistler wave propagation effects are non negligible in electron MHD turbulence, the local spectral energy transfer process is independent of the linear wave dispersion relation and the energy spectrum may be predicted by a Kolmogorov type argument. Direct numerical simulations were used to illustrate the theory but no mean magnetic field was introduced (Biskamp et al., 1996; Ng et al., 2003). Dastgeer et al. (2000) investigated the turbulence regime in the presence of a moderate background magnetic field B_0 and provided convincing numerical evidence that turbulence is anisotropic. It was argued that although whistler waves may appear to play a negligible role in determining the spectral index, they are important in setting up an anisotropic turbulent cascade. The whistler wave turbulence regime was then investigated theoretically by Galtier and Bhattacharjee (2003, 2005) in the limit $B_0 \to +\infty$ (with respect to the fluctuations). It was shown that similarly to the MHD case, anisotropy is a central feature of such a turbulence. Attempts to find an anisotropic law for electron MHD was made by Cho and Vishniac (2004) (see also, Galtier et al., 2005) and a scaling relation in $k_\parallel \sim k_\perp^{1/3}$ was found both heuristically and numerically.

Hall MHD is an extension of the standard MHD where the ion inertia is retained in Ohm's law. It provides a multiscale description of magnetized plasmas from which both standard and electron MHD approximations may be recovered. Hall MHD is often used to understand, for example, the magnetic field evolution in neutron star crusts (Goldreich and Reisenegger, 1992), the turbulent dynamo (Mininni et al., 2003), the formation of filaments (Laveder, Passot and Sulem, 2002), the multiscale solar wind turbulence (Ghosh et al., 1996; Krishan and Mahajan, 2004; Galtier, 2006a,b), or the dynamics of the magnetosheath (Belmont and Rezeau, 2001). Anisotropy in Hall MHD is clearly less understood than in MHD or electron MHD mainly because the numerical treatment is more limited since a wide range of scales is necessary to detect any multiscale effects. From a theoretical point of view,

it is only recently that a wave turbulence theory has been achieved for the incompressible case (Galtier, 2006b; Sahraoui et al., 2007). For such a turbulence, a global tendency towards anisotropy was found (with, however, a weaker effect at intermediate scales) with nonlinear transfers preferentially in the direction perpendicular to the external magnetic field $\mathbf{B_0}$. The energy spectrum is characterized by two inertial ranges, separated by a knee, which are exact solutions of the wave kinetic equations. The position of the knee corresponds to the scale where the Hall term becomes sub/dominant. To date, compressible Hall MHD is still an open problem in the regime of wave turbulence. (A first step was made by Sahraoui, Belmont and Rezeau (2003) who found the Hamiltonian system.)

1.4 Wave turbulence formalism

1.4.1 *Wave amplitude equation*

Wave turbulence is the study of the long time statistical behavior of a sea of weakly nonlinear dispersive waves. It is described by wave kinetic equations. In this section we present the wave turbulence formalism which leads to these nonlinear equations. We shall use the inviscid model equation

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = \mathcal{L}(\mathbf{u}) + \varepsilon \mathcal{N}(\mathbf{u},\mathbf{u}) \,, \tag{1.18}$$

where $\mathbf{u}$ is a stationary random vector, $\mathcal{L}$ is a linear operator which insures that waves are solutions of the linear problem, and $\mathcal{N}$ is a quadratic nonlinear operator (like for MHD-type fluids). The factor ε is a small parameter ($0 < \varepsilon \ll 1$) which will be used for the weakly nonlinear expansion. For all the applications considered here, the smallness of the nonlinearities is the result of the presence of a strong uniform magnetic field $\mathbf{B_0}$; the operator $\mathcal{L}$ is thus proportional to B_0.

We introduce the 3D direct and inverse Fourier transforms

$$\mathbf{u}(\mathbf{x},t) = \int_{\mathbf{R}^3} \mathbf{A}(\mathbf{k},t) \exp(i\mathbf{k}\cdot\mathbf{x}) d\mathbf{k} \,, \tag{1.19}$$

$$\mathbf{A}(\mathbf{k},t) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \mathbf{u}(\mathbf{x},t) \exp(-i\mathbf{k}\cdot\mathbf{x}) d\mathbf{x} \,. \tag{1.20}$$

Therefore, a Fourier transform of equation (1.18) gives for the j-component

$$\left(\frac{\partial}{\partial t} + i\omega(\mathbf{k})\right) A_j(\mathbf{k},t) = \tag{1.21}$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn} A_m(\mathbf{p},t) A_n(\mathbf{q},t) \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \,,$$

where $\omega(\mathbf{k}) = \omega_k$ is given by the appropriate dispersion relation (with in general $\omega(-\mathbf{k}) = -\omega(\mathbf{k})$) and $\mathcal{H}$ is a symmetric function in its vector arguments which

basically depends on the quadratic nonlinear operator $\mathcal{N}$. Note the use of the Einstein's notation. We introduce

$$\mathbf{A}(\mathbf{k},t) = \mathbf{a}(\mathbf{k},t)e^{-i\omega_k t}\,, \tag{1.22}$$

and obtain in the interaction representation

$$\frac{\partial a_j(\mathbf{k})}{\partial t} = \varepsilon \int_{\mathbf{R}^6} \mathcal{H}_{jmn}^{\mathbf{kpq}} a_m(\mathbf{p}) a_n(\mathbf{q}) e^{i\Omega_{k,pq}t} \delta_{k,pq} d\mathbf{p} d\mathbf{q}\,, \tag{1.23}$$

where the Dirac delta function $\delta_{k,pq} = \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$ and $\Omega_{k,pq} = \omega_k - \omega_p - \omega_q$; the time dependence in fields, $\mathbf{a}$, is omitted for simplicity. Relation (1.23) is the wave amplitude equation whose dependence in ε means that weak nonlinearities will modify only slowly in time the wave amplitude. By nature, the problems considered here (in MHD, electron and Hall MHD) involve mainly three-wave interaction processes as it is expected by the form of the wave amplitude equation. The exponentially oscillating term is essential for the asymptotic closure since we are interested in the long time statistical behavior for which the nonlinear transfer time is much greater than the wave period. In such a limit most of the nonlinear terms will be destroyed by random phase mixing and only a few of them – called the resonance terms – will survive. Before going to the statistical formalism, we note the following general properties that will be used

$$\mathcal{H}_{jmn}^{\mathbf{kpq}} = \left(\mathcal{H}_{jmn}^{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right)^*, \tag{1.24}$$

$$\mathcal{H}_{jmn}^{\mathbf{kpq}} \text{ is symmetric in } (\mathbf{p},\mathbf{q}) \text{ and } (m,n)\,, \tag{1.25}$$

$$\mathcal{H}_{jmn}^{0\mathbf{pq}} = 0\,, \tag{1.26}$$

where, *, stands for the complex conjugate.

1.4.2 *Statistics and asymptotics*

We turn now to the statistical description, introduce the ensemble average $\langle ... \rangle$ and define the density tensor for homogeneous turbulence

$$q_{jj'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}') = \langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}') \rangle\,. \tag{1.27}$$

We also assume that on average $\langle \mathbf{u}(\mathbf{x},t) \rangle = 0$ which leads to the relation $\mathcal{H}_{jmn}^{0\mathbf{pq}} = 0$. From the nonlinear equation (1.23), we find

$$\frac{\partial q_{jj'}\delta(k+k')}{\partial t} = \left\langle a_{j'}(\mathbf{k}')\frac{\partial a_j(\mathbf{k})}{\partial t}\right\rangle + \left\langle a_j(\mathbf{k})\frac{\partial a_{j'}(\mathbf{k}')}{\partial t}\right\rangle = \tag{1.28}$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}_{jmn}^{\mathbf{kpq}} \langle a_m(\mathbf{p}) a_n(\mathbf{q}) a_{j'}(\mathbf{k}') \rangle e^{i\Omega_{k,pq}t} \delta_{k,pq} d\mathbf{p} d\mathbf{q}$$

$$+$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}_{j'mn}^{\mathbf{k}'\mathbf{pq}} \langle a_m(\mathbf{p}) a_n(\mathbf{q}) a_j(\mathbf{k}) \rangle e^{i\Omega_{k',pq}t} \delta_{k',pq} d\mathbf{p} d\mathbf{q}\,.$$

A hierarchy of equations will clearly appear which gives for the third order moment equation

$$\frac{\partial \big\langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}') a_{j''}(\mathbf{k}'')\big\rangle}{\partial t} = \tag{1.29}$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn} \langle a_m(\mathbf{p}) a_n(\mathbf{q}) a_{j'}(\mathbf{k}') a_{j''}(\mathbf{k}'')\rangle e^{i\Omega_{k,pq}t} \delta_{k,pq} d\mathbf{p} d\mathbf{q}$$

$$+ \varepsilon \int_{\mathbf{R}^6} \Big\{ (\mathbf{k}, j) \leftrightarrow (\mathbf{k}', j') \Big\} d\mathbf{p} d\mathbf{q}$$

$$+ \varepsilon \int_{\mathbf{R}^6} \Big\{ (\mathbf{k}", j") \leftrightarrow (\mathbf{k}', j') \Big\} d\mathbf{p} d\mathbf{q} \,,$$

where in the right hand side the second line means an interchange in the notations between two pairs with the first line as a reference, and the third line means also an interchange in the notations between two pairs with the second line as a reference. At this stage, we may write the fourth order moment in terms of a sum of the fourth order cumulant plus products of second order ones, but a natural closure arises for times asymptotically large (see *e.g.*, Newell et al., 2001; Nazarenko, 2011; Newell and Rumpf, 2011). In this case, several terms do not contribute at large times like, in particular, the fourth order cumulant which is not a resonant term. In other words, the nonlinear regeneration of third order moments depends essentially on products of second order moments. The time scale separation imposes a condition of applicability of wave turbulence which has to be checked *in fine* (see *e.g.*, Nazarenko, 2007). After integration in time, we are left with

$$\big\langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}') a_{j''}(\mathbf{k}'')\big\rangle = \tag{1.30}$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn} \Big(\langle a_m(\mathbf{p}) a_n(\mathbf{q})\rangle \langle a_{j'}(\mathbf{k}') a_{j''}(\mathbf{k}'')\rangle + \langle a_m(\mathbf{p}) a_{j'}(\mathbf{k}')\rangle \langle a_n(\mathbf{q}) a_{j''}(\mathbf{k}'')\rangle$$

$$+ \langle a_m(\mathbf{p}) a_{j''}(\mathbf{k}'')\rangle \langle a_n(\mathbf{q}) a_{j'}(\mathbf{k}')\rangle \Big) \Delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}$$

$$+ \varepsilon \int_{\mathbf{R}^6} \Big\{ (\mathbf{k}, j) \leftrightarrow (\mathbf{k}', j') \Big\} d\mathbf{p} d\mathbf{q} + \varepsilon \int_{\mathbf{R}^6} \Big\{ (\mathbf{k}", j") \leftrightarrow (\mathbf{k}', j') \Big\} d\mathbf{p} d\mathbf{q} \,,$$

where

$$\Delta(\Omega_{k,pq}) = \int_0^{t \gg 1/\omega} e^{i\Omega_{k,pq}t'} dt' = \frac{e^{i\Omega_{k,pq}t} - 1}{i\Omega_{k,pq}} \,. \tag{1.31}$$

The same convention as in (1.29) is used. After integration in wave vectors $\mathbf{p}$ and $\mathbf{q}$ and simplification, we get

$$\big\langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}') a_{j''}(\mathbf{k}'')\big\rangle = \tag{1.32}$$

$$\varepsilon \Delta(\Omega_{kk'k''}) \delta_{kk'k''}$$

$$\Big(\mathcal{H}_{jmn}^{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} q_{mj'}(\mathbf{k}')q_{nj''}(\mathbf{k}'') + \mathcal{H}_{jmn}^{\mathbf{k}-\mathbf{k}''-\mathbf{k}'} q_{mj''}(\mathbf{k}'')q_{nj'}(\mathbf{k}')$$

$$+\mathcal{H}_{j'mn}^{\mathbf{k}'-\mathbf{k}-\mathbf{k}''} q_{mj}(\mathbf{k})q_{nj''}(\mathbf{k}'') + \mathcal{H}_{j'mn}^{\mathbf{k}'-\mathbf{k}''-\mathbf{k}} q_{mj''}(\mathbf{k}'')q_{nj}(\mathbf{k})$$

$$+\mathcal{H}_{j''mn}^{\mathbf{k}''-\mathbf{k}-\mathbf{k}'} q_{mj}(\mathbf{k})q_{nj'}(\mathbf{k}') + \mathcal{H}_{j''mn}^{\mathbf{k}''-\mathbf{k}'-\mathbf{k}} q_{mj'}(\mathbf{k}')q_{nj}(\mathbf{k})\Big)\,.$$

The symmetries (1.25) lead to

$$\left\langle a_j(\mathbf{k})a_{j'}(\mathbf{k}')a_{j''}(\mathbf{k}'')\right\rangle = \tag{1.33}$$

$$2\varepsilon\Delta(\Omega_{kk'k''})\delta_{kk'k''}\Big(\mathcal{H}_{jmn}^{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} q_{mj'}(\mathbf{k}')q_{nj''}(\mathbf{k}'')$$

$$+\mathcal{H}_{j'mn}^{\mathbf{k}'-\mathbf{k}-\mathbf{k}''} q_{mj}(\mathbf{k})q_{nj''}(\mathbf{k}'') + \mathcal{H}_{j''mn}^{\mathbf{k}''-\mathbf{k}-\mathbf{k}'} q_{mj}(\mathbf{k})q_{nj'}(\mathbf{k}')\Big)\,.$$

The latter expression may be introduced into (1.28). We take the long time limit (which introduces irreversibility) and find

$$\Delta(x) \to \pi\delta(x) + i\mathcal{P}(1/x)\,, \tag{1.34}$$

with $\mathcal{P}$ the principal value of the integral. We finally obtain the asymptotically exact wave kinetic equations

$$\frac{\partial q_{jj'}(\mathbf{k})}{\partial t} = 4\pi\varepsilon^2 \int_{\mathbf{R}^6} \delta_{k,pq}\delta(\Omega_{k,pq})\mathcal{H}_{jmn}^{\mathbf{kpq}} \tag{1.35}$$

$$\Big(\mathcal{H}_{mrs}^{\mathbf{p}-\mathbf{q}-\mathbf{k}} q_{rn}(\mathbf{q})q_{j's}(\mathbf{k}) + \mathcal{H}_{nrs}^{\mathbf{q}-\mathbf{p}\mathbf{k}} q_{rm}(\mathbf{p})q_{j's}(\mathbf{k}) + \mathcal{H}_{j'rs}^{-\mathbf{k}-\mathbf{p}-\mathbf{q}} q_{rm}(\mathbf{p})q_{sn}(\mathbf{q})\Big) d\mathbf{p}d\mathbf{q}\,.$$

These general 3D wave kinetic equations are valid in principle for any situation where three-wave interaction processes are dominant; only the form of $\mathcal{H}$ has to be adapted to the problem. Equation for the (total) energy is obtained by taking the trace of the tensor density, $q_{jj}(\mathbf{k})$, whereas other inviscid invariants are found by including non diagonal terms.

1.4.3 *Wave kinetic equations*

Equation (1.35) is the wave kinetic equation for the spectral tensor components. We see that the nonlinear transfer is based on a resonance mechanism since we need to satisfy the relations

$$\omega_k = \omega_p + \omega_q\,, \tag{1.36}$$

$$\mathbf{k} = \mathbf{p} + \mathbf{q}\,. \tag{1.37}$$

The solutions define the resonant manifolds which may have different forms according to the flow. For example in the limit of weakly compressible MHD, when the sound speed is much greater than the Alfvén speed, it is possible to find (for the shape of the resonant manifolds in the 3D k-space) spheres or tilted planes for Fast-Fast-Alfvén and Fast-Fast-Slow wave interactions, and rays (a degenerescence of the

resonant manifolds) for Fast-Fast-Fast wave interactions (Galtier et al., 2001). We also find planes perpendicular to the uniform magnetic field $\mathbf{B_0}$ for Slow-Slow-Slow, Slow-Slow-Alfvén, Slow-Alfvén-Alfvén or Alfvén-Alfvén-Alfvén wave interactions; it is a similar situation to incompressible MHD turbulence (since, at first order, slow waves have the same frequency as Alfvén waves) for which the resonant manifolds foliate the Fourier space.

The representation of the resonant manifolds is always interesting since it gives an idea of how the spectral densities can be redistributed along (or transverse to) the mean magnetic field direction whose main effect is the nonlinear transfer reduction along its direction (Matthaeus et al., 1996). The previous finding was confirmed by a detailed analysis of wave compressible MHD turbulence (Kuznetsov, 2001; Chandran, 2005) in the small β limit where the wave kinetic equations were derived as well as their exact power law solutions. The situation for electron and Hall MHD is more subtle and there is no simple picture for the resonant manifolds like in MHD. In this case, it is nevertheless important to check if the resonance condition allows simple particular solutions in order to justify the domination of three-wave interaction processes over higher order (four-wave) processes.

The form of the wave kinetic equations (1.35) is the most general one for a dispersive problem as whistler wave turbulence in electron MHD. The incompressible MHD system constitutes a unique example of pseudo-dispersive waves for which wave turbulence applies. In this particular case, some symmetries are lost and the principal value terms remain present. For that reason, incompressible MHD may be seen as a singular limit of incompressible Hall MHD (Galtier, 2006b).

1.4.4 *Finite flux solutions*

The most spectacular result of the wave turbulence theory is its ability to provide exact finite flux solutions. These solutions are found after applying to the wave kinetic equations a conformal transform proposed first by Zakharov (1965) for isotropic turbulence: it is the so-called Kolmogorov–Zakharov spectra. Because anisotropy is almost always present in magnetized plasmas, a bi-homogeneous conformal transform is more appropriate (Kuznetsov, 1972). This operation can only be performed if first one assumes axisymmetry. With this assumption the wave kinetic equations (1.35) write

$$\frac{\partial \tilde{q}_{jj'}(k_\perp, k_\parallel)}{\partial t} = 4\pi\varepsilon^2 \int_{\mathbf{R}^4} \delta_{k,pq}\delta(\Omega_{k,pq})\tilde{\mathcal{H}}^{\mathbf{kpq}}_{jmn} \tag{1.38}$$

$$\Big(\tilde{\mathcal{H}}^{\mathbf{p-q-k}}_{mrs}\tilde{q}_{rn}(q_\perp, q_\parallel)\tilde{q}_{j's}(k_\perp, k_\perp) + \tilde{\mathcal{H}}^{\mathbf{q-pk}}_{nrs}\tilde{q}_{rm}(p_\perp, p_\parallel)\tilde{q}_{j's}(k_\perp, k_\parallel)$$

$$+\tilde{\mathcal{H}}^{\mathbf{-k-p-q}}_{j'rs}\tilde{q}_{rm}(p_\perp, p_\parallel)\tilde{q}_{sn}(q_\perp, q_\parallel)\Big) dp_\perp dp_\parallel dq_\perp dq_\parallel \,,$$

where $\tilde{q}_{jj'}(k_\perp, k_\parallel) = q_{jj'}(\mathbf{k})/(2\pi k_\perp)$ and $\tilde{\mathcal{H}}$ is a geometric operator. Note that we have performed an integration over the polar angle. Except for incompressible MHD

for which the wave kinetic equations simplify thanks to the absence of nonlinear transfer along the parallel direction (along $\mathbf{B_0}$), in general we have to deal with a dynamics in the perpendicular and parallel directions with a relatively higher transfer transverse to $\mathbf{B_0}$ than along it. In this case, we may write the wave kinetic equations in the limit $k_\perp \gg k_\parallel$. The system of integro-differential equations obtained is then sufficiently reduced to allow us to extract the exact power law solutions. We perform the conformal transformation to the equations for the invariant spectral densities (total energy, magnetic helicity...)

$$\begin{aligned}
p_\perp &\to k_\perp^2/p_\perp\,, \\
q_\perp &\to k_\perp q_\perp/p_\perp\,, \\
p_\parallel &\to k_\parallel^2/p_\parallel\,, \\
q_\parallel &\to k_\parallel q_\parallel/p_\parallel\,,
\end{aligned} \tag{1.39}$$

and we search for stationary solutions in the power law form $k_\perp^{-n}k_\parallel^{-m}$. Basically two types of solutions are found: the fluxless solution, also called the thermodynamic equilibrium solution, which corresponds to the equipartition state for which the flux is zero, and the finite flux solution which is the most interesting one. During the last decades many papers have been devoted to the finding of these finite flux solutions for isotropic as well as anisotropic wave turbulence (Zakharov et al., 1992; Nazarenko, 2011).

Recently, and thanks to high numerical resolutions, a new challenge has appeared in wave turbulence: the study of incompressible Alfvén wave turbulence (Galtier et al., 2000) reveals that the development of the finite flux solution of the wave kinetic equations is preceded by a front characterized by a significantly steeper scaling law. An illustration is given in Fig. 1.3 for the balance case ($E = E^+ = E^-$): the temporal evolution of the energy spectrum reveals the formation of a $k_\perp^{-7/3}$ front before the establishment of the Kolmogorov-Zakharov solution in $k_\perp^{-2}$. This finding is in contradiction with the theory proposed by Falkovich and Shafarenko (1991) on the nonlinear front propagation where the authors claimed that the Kolmogorov–Zakharov spectrum should be formed right behind the propagating front. The same observation was also made for the inverse cascade in the nonlinear Schrödinger equation (Lacaze et al., 2001) and a detailed analysis was given by Connaughton et al. (2003) in the simplified case of strongly local interactions for which the usual wave kinetic equations become simple PDEs (for MHD, see, Galtier and Buchlin, 2010). In spite of these attempts, the anomalous scaling is still an open problem without clear physical and proper mathematical answer.

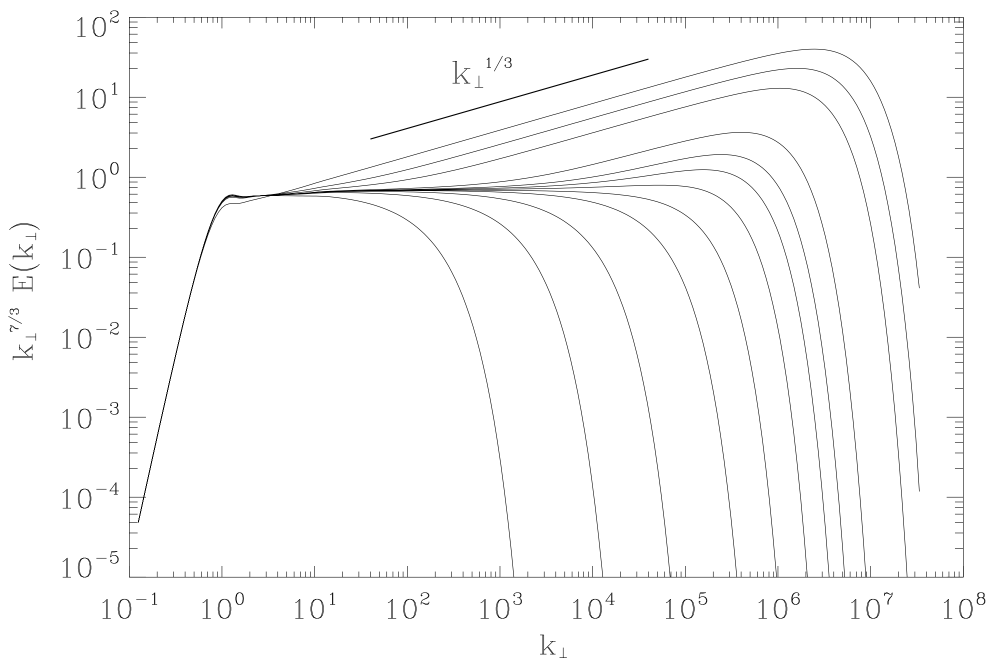

Fig. 1.3 Time propagation (from left to right) of the compensated energy spectrum for incompressible Alfvén wave turbulence ($E = E^+ = E^-$): an anomalous scaling is observed before the formation of the Kolmogorov–Zakharov spectrum.

1.5 Main results and predictions

1.5.1 *Alfvén wave turbulence*

We start to summarize the results of wave turbulence in the incompressible MHD case for which the wave kinetic equations are singular in the sense that the principal value terms remain except for the Elsässer energies $E^\pm$. As it was explained, the origin of this particularity is the pseudo-dispersive nature of Alfvén waves which are a unique case where wave turbulence theory applies (see also Zakharov and Sagdeev, 1970). In the limit of strongly anisotropic turbulence for which $k_\perp \gg k_\parallel$, the wave kinetic equations (1.38) simplify. Then, the Elsässer energy spectra of the transverse fluctuations (shear-Alfvén waves) satisfy the asymptotic integro-differential equations

$$\frac{\partial E^\pm(k_\perp, k_\parallel)}{\partial t} = \tag{1.40}$$

$$\frac{\pi \varepsilon^2}{B_0} \iint_\Delta \cos^2\phi \sin\theta \, \frac{k_\perp}{q_\perp} E^\mp(q_\perp, 0) \left[k_\perp E^\pm(p_\perp, k_\parallel) - p_\perp E^\pm(k_\perp, k_\parallel)\right] dp_\perp dq_\perp ,$$

where Δ defines the domain of integration on which $\mathbf{k}_\perp = \mathbf{p}_\perp + \mathbf{q}_\perp$, ϕ is the angle between $\mathbf{k}_\perp$ and $\mathbf{p}_\perp$ and θ the angle between $\mathbf{k}_\perp$ and $\mathbf{q}_\perp$. A fundamental property of Alfvén wave turbulence appears on equations (1.40): the evolution of the spectra $E^\pm$ is always mediated by interaction with the slow mode $q_\parallel = 0$. This is a direct consequence of the resonance conditions discussed in Section 1.3.2.3. Then, the parallel wavenumber dependence does not affect the nonlinear dynamics: in other words, there is no transfer along $\mathbf{B_0}$ and $k_\parallel$ can be treated as an external parameter. Therefore, we introduce the notation $E^\pm(k_\perp, k_\parallel) = E^\pm(k_\perp) f_\pm(k_\parallel)$ where $f_\pm$ are arbitrary functions of $k_\parallel$ given by the initial conditions and with the assumption $f_\pm(0) = 1$ we obtain

$$\frac{\partial E^\pm(k_\perp)}{\partial t} = \tag{1.41}$$

$$\frac{\pi \varepsilon^2}{B_0} \iint_\Delta \cos^2\phi \sin\theta \, \frac{k_\perp}{q_\perp} E^\mp(q_\perp) \left[k_\perp E^\pm(p_\perp) - p_\perp E^\pm(k_\perp) \right] dp_\perp dq_\perp ,$$

which describes the transverse dynamics. The exact finite flux solutions of equations (1.41) for the stationary energy spectra are

$$E^\pm(k_\perp) \sim k_\perp^{n_\pm} \tag{1.42}$$

with

$$n_+ + n_- = -4 \, . \tag{1.43}$$

Additionally, the power law indices must satisfy the condition of locality (*i.e.* the condition on the power law solutions for which the wave kinetic equations remain finite)

$$-3 < n_\pm < -1 \, . \tag{1.44}$$

It is important to recall that wave turbulence is an asymptotic theory which must satisfy a condition of application. In our case the transfer time must be significantly larger than the Alfvén time which means in terms of wavenumbers that $k_\parallel \gg \varepsilon^2 k_\perp$ (and that at small transverse scales turbulence becomes strong). At this point a comment has to made about the Kolmogorov–Zakharov solutions (1.43). These solutions imply the contribution of the wavevector $\mathbf{q}$ which is by nature a slow mode ($q_\parallel = 0$) whereas the other contributions $\mathbf{k}$ and $\mathbf{p}$ imply wave modes ($k_\parallel = p_\parallel > 0$). In Fig. 1.3 it is assumed that $E^+ = E^-$ and that the dynamics of the slow mode is given by the wave mode hence the stationary solution $n_+ = n_- = 2$. However, it may happen that the slow mode has its own dynamics which can belong to strong turbulence. Since relation (1.43) is exact, if the wave mode energy has a $-5/3$ scaling then the wave turbulence spectrum should be $-7/3$. Although this discussion was given in the original paper (Galtier et al., 2000) it was not considered seriously until the most recent direct numerical simulations (Bigot et al., 2008b; Bigot and Galtier, 2011) reveal that this situation may happen. We will come back to this point in the conclusion.

The locality of interactions (in Fourier space) is an important issue in MHD turbulence. According to some recent works in isotropic turbulence, nonlinear interactions seem to be more non local in MHD than in a pure hydrodynamics in the sense that the transfer of energy from the velocity field to the magnetic field may be a highly non local process in Fourier space (Alexakis et al., 2005). The situation is different when we deal with anisotropic turbulence: in this case interactions (between perpendicular wavevectors) are mainly local (Alexakis et al., 2007). In wave turbulence, the condition (1.44) has to be satisfied to validate the exact power law solutions and avoid any divergence of integrals in the wave kinetic equations due to nonlocal contributions. In practice, numerical simulations of the wave kinetic equations have clearly shown that the solutions (1.44) are attractive (Galtier et al., 2000).

Recently, it was realized that the wave kinetic equations found in the anisotropic limit may be recovered without the wavenumber condition $k_\perp \gg k_\parallel$ if initially only the shear-Alfvén waves were considered (Galtier and Chandran, 2006). Therefore, the finite flux solution may be extended to the entire wavenumber space which renders its detection easier. This idea was tested successfully against numerical simulations by Perez and Boldyrev (2008) who found a spectral signatures of Alfvén wave turbulence.

1.5.2 *Compressible MHD*

We turn now to the compressible regime for which two limits have been analyzed. The first case (case I) is the one for which the plasma (thermal) pressure is assumed to be small as compared to the magnetic pressure (small β limit) and where three-wave interaction processes that do not involve slow waves are neglected (Kuznetsov, 2001). In the second case (case II) for which we still have $\beta \ll 1$, slow waves are neglected and a constant density is imposed (Chandran, 2005). In both situations the general finite flux solutions are not obvious to express.

In case I, when only interactions between Alfvén and slow waves are kept, a wave energy spectrum in $\sim k_\perp^{-2} k_\parallel^{-5/2}$ is found which corresponds to a (finite) constant energy flux solution. It is claimed that the addition of the interactions with the fast waves will lead to the same solution since the dynamics tends to produce strongly anisotropic distributions of the waves concentrated in k-space within a narrow-angle cone in the $\mathbf{B_0}$ direction ($k_\perp \ll k_\parallel$). Under these conditions, the fast waves coincide with the Alfvén waves.

In case II, the general wave kinetic equations do not allow us to find exact power law solutions. However, when only Fast-Fast-Fast interactions are kept it is possible to find a finite flux solution for the fast wave 1D energy spectrum which scales as $\sim f(\theta) k^{-3/2}$, where $f(\theta)$ is an arbitrary function of the angle θ between the wave vector $\mathbf{k}$ and the uniform magnetic field $\mathbf{B_0}$. Numerical simulations of the general wave kinetic equations are made to find the behavior according to the angle θ. A

solution close to $k_\perp^{-2}$ is found for the Alfvén wave 2D energy spectrum (for different fixed $k_\parallel$) when $k_\perp \gg k_\parallel$. The k–spectra plotted at $\theta = 45^o$ reveal a fast wave modal energy spectrum in $\sim k^{-3/2}$ and an steeper Alfvén wave spectrum, while for a small angle ($\theta = 7.1^o$) both spectra follow the same scaling law steeper than $k^{-3/2}$ (Chandran, 2005).

1.5.3 *Whistler wave turbulence*

The electron MHD equations in the presence of a strong uniform magnetic field B_0 exhibit dispersive whistler waves. The wave turbulence regime was analyzed in the incompressible case by Galtier and Bhattacharjee (2003, 2005) who derived the wave kinetic equations by using a complex helicity decomposition. The strong anisotropic ($k_\perp \gg k_\parallel$) finite flux solutions correspond to

$$E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2} \tag{1.45}$$

for the magnetic energy spectrum, and

$$H(k_\perp, k_\parallel) \sim k_\perp^{-7/2} k_\parallel^{-1/2} \tag{1.46}$$

for the magnetic helicity spectrum. As for the other cases presented above, a direct cascade was found for the energy. In particular it was shown that contrary to MHD, the wave kinetic equations which involve three-wave interaction processes are characterized by a nonlinear transfer that decreases linearly with $k_\parallel$; for $k_\parallel = 0$, the transfer is exactly null. Thus the 2D modes (or slow modes) decouple from the three-dimensional whistler waves. Such a decoupling is found in a variety of problems like rotating turbulence (Galtier, 2003; Sagaut and Cambon, 2008).

1.5.4 *Hall MHD*

The last example exposed in this chapter is the Hall MHD case which incorporates both the standard MHD and electron MHD limits. This system is much heavier to analyze in the regime of wave turbulence and it is only recently that a theory has been proposed (Galtier, 2006b). The general theory emphasizes the fact that the large scale limit of standard MHD becomes singular with the apparition of a new type of terms, the principal value terms. Of course, the large scale and small scale limits tend to the appropriate theories (MHD and electron MHD wave turbulence), but in addition it is possible to describe the connection between them at intermediate scales (scales of the order of the ion inertial length d_i). For example, moderate anisotropy is predicted at these intermediate scales whereas it is much stronger for other scales. It is also interesting to note that the small scale limit gives a system of equations richer than the pure electron MHD system (Galtier and Bhattacharjee, 2003) with the possibility to describe the sub-dominant kinetic energy dynamics. An exact power law solution for the kinetic energy spectrum is found for ion cyclotron

wave turbulence which scales as

$$E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2} . \tag{1.47}$$

To date, the wave kinetic equations of Hall MHD have not been simulated numerically even in their simplified form (when helicity terms are discarded). It is an essential step to understand much better the dynamics at intermediate scales.

1.6 Conclusion and perspectives

1.6.1 *Observations*

Waves and turbulence are two fundamental ingredients of many space plasmas. The most spectacular illustration of such characteristics is probably given by the observations of the Sun's atmosphere with the orbital solar observatory Hinode/JAXA and more recently by SDO/NASA launched in 2010. For the first time, detection of Alfvén waves is made through the small oscillations of many thin structures called threads (see Fig. 1.4). In the meantime the highly dynamical nature of coronal loops is revealed by non-thermal velocities detected with spectrometers. These findings are considered as a remarkable step in our understanding of the solar coronal dynamics. Nowadays, it is believed that Alfvén waves turbulence is a promising model to understand the heating of the solar corona (van Ballegooijen et al., 2011).

The interplanetary medium is another example of magnetized plasma where waves and turbulence are detected. In this framework, the origin of the so-called "dissipative range", *i.e.* the extension of the turbulent inertial range beyond a fraction of hertz, is currently one of the main issue discussed in the community. Although a final answer is not given yet, wave turbulence is a promising regime to understand the inner solar wind dynamics in the sense that it gives exact results in regards to the possible multiscale behavior of magnetized plasmas as well as the intensity of the anisotropic transfer between modes.

The main feature of magnetized plasmas in the regime of wave turbulence is the omnipresence of anisotropy and the possibility to have different spectral scaling laws according to the space direction. To achieve a proper comparison between observational data and theoretical predictions, not only *in situ* measurements are necessary, but multipoints data have also to be accessible. It is at this price that the true nature of magnetized plasmas will be revealed. The magnetosphere is the first medium where it is possible to perform such a comparison thanks to the Cluster/ESA mission (Sahraoui et al., 2006). In the case of the Jupiter's magnetosphere the large scale magnetic field is relatively strong. Actually, the first indirect evidence of Alfvén wave turbulence was reported in Saur et al. (2002) by using five years set of Galileo/NASA spacecraft magnetic field data. (Since the work was based on a one point analysis a model was used to differentiate the perpendicular and parallel spectra.)

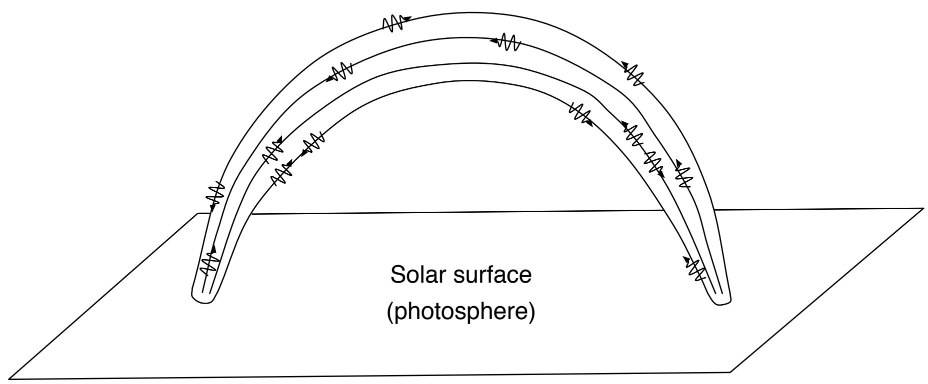

Fig. 1.4 Schematic view of Alfvén wave turbulence on the Sun: coronal loops act as a resonant cavity for Alfvén wave packets which are transmitted from the deeper layers of the solar atmosphere.

Perhaps the first direct evidence of Alfvén wave turbulence comes from solar (photospheric) magnetic field observations above active regions. Using data from SOHO/ESA, Abramenko (2005) observed that the magnetic energy spectra reduced to its radial component (along $\mathbf{B_0}$) have a transverse scaling significantly different from the Kolmogorov one with power laws generally between -2 and -2.3. Since the magnetic field can only be detected in a thin layer (the temperature is too high in the chromosphere and in the corona to measure the magnetic field with the Zeeman effect), the magnetic spectra can be interpreted as a measure at a given $k_\parallel > 0$ (in practice a small average is made in $k_\parallel$ since the data comes from slightly different altitudes). As we know from the Alfvén wave turbulence theory the parallel fluctuations follow the same dynamics as the perpendicular (shear-Alfvén) fluctuations and the addition of the power law indices of the slow and wave mode spectra is -4. Then, these unexpected spectra may be seen as the direct manifestation of Alfvén wave turbulence if the slow mode contribution scales like Kolmogorov (see the discussion on Section 1.5.1). If this interpretation is correct it is probably the most important achievement of the Alfvén wave turbulence theory.

1.6.2 *Simulations*

Numerical simulation is currently the main tool to improve our knowledge on wave turbulence in magnetized plasmas since we are still limited by the observational (single point) data. Two types of simulations are available: the simulation of the wave kinetic equations and the direct numerical simulation of the original MHD-type equations. In the former case, it is a way to find for instance the spectral scaling laws when the wave kinetic equations are too complex to provide us the exact solutions after application of the usual conformal transform. An example is given by compressible MHD: the numerical simulations revealed a variation of the

power law energy spectrum with the angle between $\mathbf{k}$ and $\mathbf{B_0}$ (Chandran, 2005). Simulations may also be useful to investigate wave turbulence when an external forcing is applied like in incompressible MHD (Galtier and Nazarenko, 2008).

For direct numerical simulation, the challenge is slightly different: indeed, in this case the main goal is the measure of the transition between strong and wave turbulence, and thus between isotropic and anisotropic turbulence. The former regime has been extensively studied since more than three decades whereas the latter is still a young subject. The main topic of such a simulation is also to find general properties that could help us to understand the single point measurements made in natural plasmas. We arrive here at the heart of current issues in wave turbulence. One of the most important points emphasized by recent direct numerical simulations in incompressible MHD is the coexistence of wave and strong turbulence (Bigot et al., 2008b). This characteristic should not be a surprise since basically wave turbulence is a perturbative theory which must satisfy conditions of applicability. In this case, the slow mode ($q_\parallel = 0$) may evolve differently from the wave modes ($k_\parallel, p_\parallel > 0$) since the former case may be characterized by strong turbulence and the latter by wave turbulence. This distinction is fundamental and not really taken into account in the community: Alfvén wave turbulence will not be fully revealed in space as well as simulated plasmas as long as strong and wave turbulence will not be separated. This point is illustrated with direct numerical simulations (Bigot et al., 2008b; Bigot and Galtier, 2011) where steeper energy spectra may be found for the wave modes (between -2 and $-7/3$) compared to the slow mode (down to $-5/3$). The result seems to depend on the relative intensity of B_0: a strong magnetic background tends to increase the energy of the slow mode which, then, may evolve independently (the critical value of B_0 seems to be around 10 times the *r.m.s.* fluctuations). It is also found that the wave turbulence spectra are generally visible at a fixed $k_\parallel > 0$ and may be hidden in a $-5/3$ scaling if a summation over $k_\parallel$ is performed (this result could be due to the limited space resolution). The same investigation clearly shows that – as expected– an equipartition between the kinetic and magnetic energies is reached in the wave modes whereas the magnetic energy is dominant in the slow mode. Note that this Alfvén wave turbulence regime was also numerically found independently by Perez and Boldyrev (2008).

1.6.3 *Open questions*

In the light of such recent results the following new questions may be addressed:

- Does the $-7/3$ spectrum more universal than -2 in Alfvén wave turbulence ?
- Is the slow mode ($q_\parallel = 0$) intermittent ? and then does it produce intermittency in the slave wave modes ($k_\parallel, p_\parallel > 0$) ?
- Can we find direct signatures of Alfvén wave turbulence in solar magnetic loops with three-dimensional magnetic field data ?

• Does Alfvén wave turbulence the universal regime for stellar magnetic loops ?

• Does the $-5/3$ energy spectrum observed in the solar wind correspond to a bias ? and then what is the wave vector energy spectrum ?

• Is the small energy ratio $E^u/E^b \sim 0.5$ found in the local solar wind (Bruno and Carbone, 2005) due essentially to the slow mode contribution in which case the wave modes may follow equipartition ?

• Are the absence of intermittency and the -2.5 magnetic spectrum observed in the solar wind dispersive regime (Kiyani et al., 2009) the signatures of whistler wave turbulence ?

Future missions like *Solar Orbiter* from the European Space Agency (scheduled for 2017) will certainly help us to answer these questions and make significant advances on plasma turbulence.

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